Continuum Modeling of Flexible Structures with Application to Vibration Control

W. H. Bennett* and H. G. Kwatny†
Techno-Sciences, Inc., Greenbelt, Maryland

In this paper we consider the construction of hybrid models for flexible structures, including both the natural spatially distributed dynamics of low-mass density structures and the lumped models associated with rigid-body dynamics. For the purposes of active control of structural vibrations we argue that by using transform methods, the standard approach to transfer function (or mechanical impedance) modeling can be extended to provide a complete formulation of a hybrid model with localized control. We proceed to give a systematic procedure for obtaining the required transfer functions and Green's functions for the hybrid model dynamics consisting of one-dimensional elastic elements (such as beams, rods, and cables) together with rigid-body (lumped) models. We focus on a well-known family of models for elastic elements together with various reductions and indicate the construction of well-posed state-space models for these systems. Using the resulting state-space, we next demonstrate the construction of a complete frequency domain characterization of the elemental system response using transfer functions and Green's functions. Hybrid constructions are discussed for various interconnections of distributed and lumped elements, and a simple illustrative example is given. We highlight the extent to which the required computations can be performed automatically on a digital computer using a computer algebra system.

Nomenclature

A	= cross-sectional area
E	= modulus of elasticity
I	= moment of inertia
L	= beam length
z	= longitudinal coordinate
κG	= effective shear modulus
ρ	= mass density
$\eta(z)$	= lateral displacement
ϕ	= angular rotation of cross section

I. Introduction

IT is now generally accepted that large, low-mass density structures will be essential for near-term space applications. Moreover, it is apparent that active control of structural vibrations will be necessary to enhance their stiffness and damping properties.[1] In this paper we consider the construction of mathematical models for elastic dynamics of space structures suitable for the design of active vibration control laws for these systems.

The success of active control for such structures will hinge to some extent on the ability of a control law to react to vibratory responses that may be initially localized before they propagate throughout a structure. This leads naturally to questions of how to implement active control so as to distribute the control effort spatially as needed. We contend that existing methods for the design of control systems for distributed parameter systems can be applied effectively if appropriate continuum models for the candidate space structures can be computed. The nature of the required models, however, can be quite different from those obtained by the standard finite-element methods that are popular for large structural analysis problems throughout the aerospace industry. In this paper we delineate a method for constructing the (possibly irrational) transfer functions and Green's functions required to describe the distributed dynamics of interconnected structures for control system design. The methods we employ focus on the construction of exact algebraic expressions for the required functions. This approach is significant in that numerical issues arising in the evaluation of the irrational models of the system frequency responses can be addressed in the context of the control problem and can be consistent with the control objective. Although a considerable literature exists on the computation of Green's functions for interconnected systems with both distributed and lumped parameter elements[2] and with specific application to interconnected mechanical structures,[3,4] application of these techniques for modeling aerospace structures has been limited. The contribution of the results contained in this paper serves primarily to focus these methods on the construction of models required for design of optimal distributed parameter control of a class of flexible space structures.

We begin in this section with a brief overview of continuum models for active structural control. One method for the computation of a distributed control law for a distributed parameter model then is briefly summarized. This discussion serves to motivate our interest in the continuum modeling methods considered in the sequel.

Comprehensive models of flexible spacecraft dynamics will involve systems with fairly complex interconnections of lumped and distributed subsystems, and therefore we intend to construct the overall models by first developing subsystem models and then combining them according to the required interconnection rules. In Sec. II, basic questions of causality and well-posedness of certain standard models for beams are reviewed. These equations are crucial to the computation of hybrid state-space models discussed in Sec. III for an integrated structural system. Throughout this effort we have focused on the potential for model construction using computer-aided computation, a combination of modern computer algebra (symbolic manipulation) and numerical methods. In our efforts we have used the computer algebra system program SMP.[5,6]

A. Generic Control Problem for Flexible Structures

In this section we consider a commonly used generic model for elastic dynamics of a spatially continuous structure. We

Presented as Paper 86-0173 at the AIAA 24th Aerospace Sciences Meeting, Reno, NV, Jan. 6-9, 1986; received April 20, 1987; revision received June 10, 1988.

*Senior Research Engineer. Member AIAA.
†Senior Scientist; and Professor of Systems Engineering, Drexel University, Philadelphia, PA.

summarize the construction of a state-space model and introduce a typical control problem for vibration suppression. We highlight the modal approximations that are popular for these problems and proceed to demonstrate an alternate technique for model construction and control design. Modeling and control law computation can proceed in the frequency domain, based on transfer function methods. We focus on the class of structural control problems for which the question of control of propagation of wavelike disturbances is important.

A popular continuum model for a flexible structure[1,7-9] is described by a system of partial differential equations (PDE):

$$m(z)\frac{\partial^2 w(t,z)}{\partial t^2} + D_0\frac{\partial w(t,z)}{\partial t} + A_0 w(t,z) = F(t,z) \qquad (1)$$

where $w(t,z)$ is an N vector of displacements of a structure Ω with respect to some equilibrium for Ω is a bounded, open set in $\boldsymbol{R}^N$. The vector $z \in \Omega$ is a coordinate in Ω. We assume the boundary $\partial\Omega$ is smooth. The mass density $m(z)$ is positive definite and bounded on $\partial\Omega$. The damping term $D_0\partial w/\partial t$ models both gyroscopic (asymmetric) and structural (symmetric) damping effects. The internal restoring force $A_0 w$ is generated by a time-invariant, differential operator A_0 for the structure. For most common structural models, A_0 is an unbounded differential operator with domain $D(A_0)$ consisting of certain smooth functions satisfying appropriate boundary conditions on $\partial\Omega$. Thus, for these problems, $D(A_0)$ is typically dense in the Hilbert space $\mathcal{H}_0 = \mathcal{L}_2(\Omega)$ endowed with its natural inner product $\langle x,y\rangle_0 = \int_\Omega x^T(z)y(z)\,dz$. Often (but not always), the spectrum of A_0, $\sigma(A_0)$, consists of discrete eigenvalues with associated eigenfunctions that constitute a basis for $\mathcal{L}_2(\Omega)$.

The applied force distribution $F(t,z)$ can be thought of as consisting of three components:

$$F(t,z) = F_d(t,z) + F_c(t,z) + F_a(t,z) \qquad (2)$$

where F_d is the N vector of exogenous disturbances (possibly forces and torques), F_c is a continuous, distributed, controlled force field (an available option only in some special applications), and F_a represents controlled forces due to localized actuation:

$$F_a(t,z) = \sum_{j=1}^{k} b_j(z)u_j(t) = B_0 u(t) \qquad (3)$$

The actuator influence functions $b_j(z)$ are highly localized in Ω and can be approximated by Dirac delta functions. We assume that a finite number p of measurements can be made as

$$y(t) = C_0 w + C_0' \frac{\partial w}{\partial t} \qquad (4)$$

where $y(t)$ is a p vector. The operators $B_0 : \boldsymbol{R}^m \to \mathcal{H}_0$, $C_0 : \mathcal{H}_0 \to \boldsymbol{R}^p$, and $C_0' : \mathcal{H}_0 \to \boldsymbol{R}^p$ are bounded.

The standard vibration control problem for this model is to find the controls $u_j(t), j = 1,\ldots,k$ (we ignore the possibility of F_c) given the observations $y(t)$ to maintain the system state, e.g.,

$$x(t,z) = \begin{bmatrix} w(t,z) \\ \dot{w}(t,z) \end{bmatrix} \qquad (5)$$

as close to its equilibrium state as possible.

The choice of state-space given by Eq. (5) is often made for models in the generic form of Eq. (1). (We will discuss later how attractive alternate state-space models can arise in hybrid constructions.) A natural assumption for structural problems[1] is that A_0 is self-adjoint with a compact resolvent and discrete (real) spectrum that is bounded from below. The state equation, Eq. (5), then can be considered as an element of a Hilbert space $\mathcal{H} = D(A_0^{1/2}) \times \mathcal{H}_0$ with the energy norm

$$\|x\|_E^2 = \langle w, A_0 w\rangle_0 + \langle m\dot{w}, \dot{w}\rangle_0 \qquad (6)$$

where the first term represents potential energy and the second term is kinetic energy. Thus, the (abstract) *state-space model* can be written as

$$\dot{x}(t,z) = Ax(t,z) + Bu(t) \qquad (7)$$

$$y(t) = Cx(t,z)$$

where

$$A = \begin{bmatrix} 0 & I \\ -A_0 & -D_0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \quad C = [C_0, C_0'] \qquad (8)$$

For transient disturbances, the standard vibration control problem is essentially an optimal regulator problem whose solution is a linear state feedback that minimizes the performance index (parametrized by a real scalar $\epsilon > 0$)

$$J(u) = \int_0^\infty (\|y\|^2 + \epsilon\|u\|^2)\,dt \qquad (9)$$

where $\|\cdot\|$ is the Euclidean norm on the appropriate finite dimensional space. This is the generic control problem surveyed in Balas.[9] In this paper we will concentrate on the construction of state-space models and the computational aspects of equations of the form of Eq. (1) and of optimal (discrete) controls $u_j(t)$ appearing in Eq. (3).

Various methods are available for approximation of the system of Eq. (7).[10,11] One method is based on a modal (eigen) expansion of A that generates a sequence of finite dimensional subspaces $\mathcal{H}_k \subset \mathcal{H}$, $k = 1,2,\ldots$, where $\mathcal{H}_k = \text{span}\{\phi_j, j = 1,\ldots,k\}$ and the $\phi_j(z)$ are eigenfunctions (or mode shapes) for A.[7,12] Based on this approximation, a sequence of finite dimensional models for Eq. (5) can be generated:

$$\dot{x}^{(k)}(t) = A^{(k)}x^{(k)}(t) + B^{(k)}u(t) \qquad (10)$$

Using a truncated model of Eq. (10) with k finite and the performance index $J(u)$ projected onto the (finite dimensional) space $\mathcal{H}_k$, one can solve the associated optimal control problem for the first k modes of A. This is a classical approach and encompasses Ritz-Galerkin methods[7] as well as spline methods.[10,11] However, as noted in Balas,[9] in all but a few special cases, the control law when applied to the system of Eq. (5) will excite higher-order modes. The inherent robustness and stability properties as well as the degree of suboptimality of control laws based on such truncated modal approximations has received a great deal of attention in both the engineering and mathematics literature.[1,9] Various alternate approaches are available to deal directly with the infinite dimensional control problem given by Eqs. (8) and (9), at least abstractly (cf., Russell[13]). One method suggested by Davis and Barry[14] and Davis and Dickinson[15] offers the advantage of a computational procedure for approximating the true optimal control in terms of the required control bandwidth. The method is based on an extension of a Weiner-Hopf solution[15] for the infinite dimensional control problem.

B. Wiener-Hopf Control

In the context of the regulator control problem given by the minimization of Eq. (9) subject to the infinite dimensional model (7), spectral factorization can be seen to provide an alternate solution[16] to a Riccati operator equation. Davis and Dickinson[14] and Davis and Barry[15] have explored the application of spectral factorization for control design of a class of distributed parameter models for long trains with multiple locomotives. Also, in Bennett and Barkakati[17] considerations for application of these results to flexible structures were given. The details of this method are outside the scope of this paper. However, in this section we will summarize the relevant results and highlight their significance for modeling of flexible structures.

Taking Laplace transforms in Eq. (7) allows one to write (at least formally)

$$\hat{Y}(s,z) = CR(s;A)x(0,z) + CR(s;A)B\hat{U}(s) \tag{11}$$

The transfer function is $G(s) = CR(s;A)B$, where $R(s;A)$ is the resolvent operator for A, $R(s;A): \mathcal{H} \to \rho(A) \subset \mathcal{H}$, where $\rho(A)$ is the resolvent set (or compliment to the spectrum of A).

The optimal control law that minimizes Eq. (9) subject to Eq. (7) consists of (linear) state feedback

$$u(t) = -B^*Kx(t,z) \tag{12}$$

where B^* is the formal adjoint of B in $\mathcal{H}$. We remark that B^*K is a (linear) integral operator on $\mathcal{H}$ that can be computed exactly without recourse to an infinite dimensional Riccati operator equation via the formula[15]

$$B^*K = \frac{1}{2\pi}\int_{-\infty}^{\infty} [F^*(i\omega)]^{-1}G^*(i\omega;A)CR(i\omega;A)\,\mathrm{d}\omega \tag{13}$$

where $F(s)$ is the unique, causal spectral factor of

$$I + G^T(-s)G(s) = F(s)F^T(-s) \tag{14}$$

For the infinite dimensional system (7), the transfer functions $G(s)$, $F(s)$ are irrational and $F(s)$ [resp. $F^T(-s)$] is analytic for $Re\ s > 0$ (resp. $Re\ s < 0$). Computational algorithms for Eq. (14) are given in Davis[15] and Bennett[17] where a numerical algorithm is given.

In this framework, the question of the modeling of flexible structures for the design of feedback control for the suppression of (linear) vibrations centers on the computation of 1) the irrational transfer function $G(s)$, and 2) the resolvent $R(s;A)$. Then spectral factorization in Eq. (14) is performed using the Davis-Dickinson algorithm,[15] and a numerical approximation to Eq. (13) can be obtained that is valid in the frequency bandwidth of the desired control action. We remark that, although all computations are in the frequency domain, the resulting control law is a linear, distributed state feedback. In the rest of this paper we focus on the modeling problem for standard elastic structural components consistent with these requirements.

II. State-Space Models for Distributed Elements

A. Standard Forms for Linear PDE's

In this section we will consider the problem of deriving a standard state-space description for typical distributed elements. In the next section we use these descriptions to compute the complete frequency domain response as required for the control problem outlined earlier. Systems of interest to us—specifically beams with one space variable and perhaps several degrees of freedom—can be represented by one of two standard forms. Once the structure of these standard models is identified, it is straightforward, although far from trivial, to mechanize the construction of the required transfer matrices using symbolic computation.[17] Moreover, in order to assemble hybrid system models by the interconnection of components or subsystems, it is essential to have a clear understanding of the causal requirements of the component mathematical models. The following paragraphs develop the required concepts in terms of commonly used structural elements. Since typical elements interact at physical boundaries, our foremost concern is with the formulation of appropriate boundary conditions for well-posed initial-boundary value problems.

Before proceeding, we recall some classical terminology associated with systems of partial differential equations. Consider the system of first-order, partial differential equations defined for $t \geq 0$ and $0 \leq z \leq L$:

$$E\frac{\partial w}{\partial t} = \tilde{F}\frac{\partial w}{\partial z} + \tilde{H}w, \quad w \in R^n \tag{15}$$

If E is nonsingular, then Eq. (15) can be written as

$$\frac{\partial w}{\partial t} = F\frac{\partial w}{\partial z} + Hw \tag{16}$$

where $F = E^{-1}\tilde{F}$, $H = E^{-1}\tilde{H}$. If F has only real eigenvalues and a complete set of eigenvectors, then the system is said to be *hyperbolic* (see, e.g., Zauderer[18]). If there are multiple real eigenvalues and less than a complete set of eigenvectors, then the system is of (partial) *parabolic* type. If all of the eigenvalues are complex, the system is of *elliptic* type. Systems with complex eigenvalues are not causal. Lyczkowski et al.[19] and Sursock[20] provide an interesting discussion of this point in connection with a fluid flow problem. The underlying problem is that systems with complex eigenvalues are not well posed as initial value problems.[21,22] We will not consider such problems any further.

If E is singular, Eq. (15) can give rise to mixed systems of all types. Some typical examples can be found in Friedly[23] and Lapidus and Pinder.[24] Our interest in this case will be limited to purely parabolic systems of the type

$$\frac{\partial w}{\partial t} = G\frac{\partial^2 w}{\partial z^2} + F\frac{\partial w}{\partial z} + Hw \tag{17}$$

which commonly arise in engineering problems.

In addition to Eqs. (16) or (17), there are associated initial and boundary conditions. For Eq. (16), these conditions take the general form

Initial conditions:

$$w(z,0) = f(z) \tag{18a}$$

Boundary conditions:

$$\Sigma_1 w(0,t) + \Gamma_1 w(L,t) = g(t) \tag{18b}$$

where $dim(g) = dim(w)$, and for Eq. (17), they take the general form

Initial conditions:

$$w(z,0) = f(z) \tag{19a}$$

Boundary conditions:

$$\Sigma_1 w(0,t) + \Sigma_2\frac{\partial w}{\partial z}(0,t) + \Gamma_1 w(L,t) + \Gamma_2\frac{\partial w}{\partial z}(L,t) = g(t) \tag{19b}$$

where $dim(g) = 2dim(w)$.

It is well known that the coefficient matrices in Eqs. (18) and (19) must satisfy certain constraints if the problem formulation is to be well posed. In the hyperbolic case [Eqs. (16) and 18)], these constraints essentially require that the boundary conditions be compatible with the wave directions. Further discussion can be found in Russell[13] and Agarwala.[11] In the following sections we will discuss some standard models for beams. The following notation is standard and assumes that the elastic beam is uniform (i.e., the parameters are independent of the spatial coordinate).

1. *Timoshenko Beam Model*

We will show how some conventional beam models can be reduced to the standard forms described in the preceding paragraphs. In particular, we will begin with the Timoshenko model and then consider two commonly used approximations that can be derived from it, the Bernoulli-Euler model and the "string" model. The equations of motion can be derived using the Lagrange equations[25] and in the absence of dissipation take the form

$$\rho A\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial z}\left[\kappa GA\left(\frac{\partial \eta}{\partial z} - \phi\right)\right] \tag{20a}$$

$$\rho I \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial z}\left[EI \frac{\partial \phi}{\partial z}\right] + \kappa GA\left(\frac{\partial \eta}{\partial z} - \phi\right) \tag{20b}$$

along with the natural boundary conditions for $\alpha = 0,L$:
Displacement:

$$\eta(\alpha,t) = \eta_\alpha t \tag{21a}$$

Shear force:

$$\kappa GA\left[\frac{\partial \eta(\alpha,t)}{\partial z} - \phi(\alpha,t)\right] = f_\alpha(t) \tag{21b}$$

Rotation:

$$\phi(\alpha,t) = \phi_\alpha t \tag{22a}$$

Moment:

$$EI\left[\frac{\partial \phi(\alpha,t)}{\partial z}\right] = \tau_\alpha(t) \tag{22b}$$

Equations (20) can be replaced by four first-order equations by introducing two new variables, $\nu(z,t)$ and $\gamma(z,t)$:

$$\frac{\partial \eta}{\partial t} = \frac{\partial \nu}{\partial z} \tag{23a}$$

$$\frac{\partial \nu}{\partial t} = \frac{\kappa G}{\rho}\left(\frac{\partial \eta}{\partial z} - \phi\right) \tag{23b}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \gamma}{\partial z} + \frac{A}{I}\nu \tag{23c}$$

$$\frac{\partial \gamma}{\partial t} = \frac{E}{\rho}\frac{\partial \phi}{\partial z} \tag{23d}$$

These equations clearly represent a hyperbolic system, and the natural boundary conditions become, for $\alpha = 0,L$,
Displacement:

$$\eta(\alpha,t) = \eta_\alpha(t) \tag{24a}$$

Shear force:

$$\nu(\alpha,t) = \nu_\alpha(t), \dot{\nu}_\alpha(t) = \frac{f_\alpha(t)}{\rho A} \tag{24b}$$

Rotation:

$$\phi(\alpha,t) = \phi_\alpha(t) \tag{25a}$$

Moment:

$$\gamma(\alpha,t) = \gamma_\alpha(t), \dot{\gamma}_\alpha(t) = \frac{\tau_\alpha(t)}{\rho I} \tag{25b}$$

Note that the boundary conditions applied to the first-order system, (23) require the time integral of boundary forces or moments applied to the beam. It is easy to confirm that the transfer functions relating forces or moments to displacements or rotations as derived from either Eqs. (20) or (23) are indeed identical and that the required integration of the shear force or moment is essential in the first-order forms.

2. *Bernoulli-Euler Beam Model*

The Bernoulli-Euler model is obtained from the Timoshenko model with two additional assumptions: 1) rotational inertia is neglected, $\rho I \to 0$; and 2) shear deformation is neglected, $(\partial\eta/\partial z) - \phi \to 0$. The first assumption reduces Eq. (20b) to

$$\kappa GA\left(\frac{\partial \eta}{\partial z} - \phi\right) = -\frac{\partial}{\partial z}\left(EI \frac{\partial \phi}{\partial z}\right) \tag{26}$$

Equation (26) and the second assumption now are used to reduce Eq. (20a) to

$$\rho A \frac{\partial^2 \eta}{\partial t^2} = -\frac{\partial^2}{\partial z^2}\left(EI \frac{\partial^2 \eta}{\partial z^2}\right) \tag{27}$$

Note that Eq. (26) along with the second assumption leads to the following expression for shear force:

$$f = \kappa GA\left(\frac{\partial \eta}{\partial z} - \phi\right) = -\frac{\partial}{\partial z}\left(EI \frac{\partial^2 \eta}{\partial z^2}\right) \tag{28}$$

The boundary conditions, Eqs. (21) and (22), reduce, for $\alpha = 0,L$, to
Displacement:

$$\eta(\alpha,t) = \eta_\alpha(t) \tag{29a}$$

Shear force:

$$-EI \frac{\partial^3 \eta(\alpha,t)}{\partial z^3} = f_\alpha(t) \tag{29b}$$

Displacement:

$$\frac{\partial \eta(\alpha,t)}{\partial z} = \phi(\alpha,t) = \phi_\alpha(t) \tag{30a}$$

Moment:

$$EI \frac{\partial^2 \eta(\alpha,t)}{\partial z^2} = \tau_\alpha(t) \tag{30b}$$

Equation (27) can be reduced to "first-order" form by introducing a new variable $\gamma(z,t)$:

$$\frac{\partial \eta}{\partial t} = -\frac{I}{A}\frac{\partial^2 \gamma}{\partial z^2} \tag{31a}$$

$$\frac{\partial \gamma}{\partial t} = -\frac{E}{\rho}\frac{\partial^2 \eta}{\partial z^2} \tag{31b}$$

and the boundary conditions associated with Eqs. (31) are, for $\alpha = 0,L$,:
Displacement:

$$\eta(\alpha,t) = \eta_\alpha(t) \tag{32a}$$

Shear force:

$$\frac{\partial \gamma(\alpha,t)}{\partial z} = \gamma_\alpha'(t), \ \dot{\gamma}_\alpha' = -\frac{A}{I} f_\alpha(t) \tag{32b}$$

Rotation:

$$\frac{\partial \eta(\alpha,t)}{\partial z} = \phi_\alpha(t) \tag{33a}$$

Moment:

$$\gamma(\alpha,t) = \gamma_\alpha(t), \ \dot{\gamma}_\alpha = \frac{\tau_\alpha(t)}{\rho I} \tag{33b}$$

Observe that Eqs. (31) are a parabolic system of the type of Eq. (17). Equations (31-33) can be derived directly from Eq. (27) or Eqs. (23) upon invoking the first and second assumptions. We also should note that a corresponding expression for shear force becomes

$$f = -\left(\frac{I}{A}\right)\frac{\partial^2 \gamma}{\partial t \partial z} \tag{34}$$

3. "String" Model

In some situations, bending deformation may be negligible with respect to shear deformation, i.e., $|\phi| \ll |\partial\eta/\partial z|$. In this case, Eq. (20a) reduces to

$$\rho A \frac{\partial^2\eta}{\partial t^2} = \frac{\partial}{\partial z}\left(\kappa GA \frac{\partial\eta}{\partial z}\right) \tag{35}$$

with the following boundary conditions for $\alpha = 0,L$:

Displacement:

$$\eta(\alpha,t) = \eta_\alpha(t) \tag{36a}$$

Shear force:

$$\kappa GA \frac{\partial\eta(\alpha,t)}{\partial z} = f_\alpha(t) \tag{36b}$$

This simple model is primarily useful for illustrative purposes. Again, by introducing the new variable $\nu(z,t)$, Eq. (35) can be replaced by the following two first-order equations:

$$\frac{\partial\eta}{\partial t} = \frac{\partial\nu}{\partial z} \tag{37a}$$

$$\frac{\partial\nu}{\partial t} = \frac{\kappa G}{\rho}\frac{\partial\eta}{\partial z} \tag{37b}$$

subject to the following boundary conditions for $\alpha = 0,L$:

Displacement:

$$\eta(\alpha,t) = \eta_\alpha \tag{38a}$$

Shear force:

$$\nu(\alpha,t) = \nu_\alpha(t), \; \dot\nu_\alpha = \frac{f_\alpha(t)}{\rho A} \tag{38b}$$

B. Beam Models with Dissipation

Various dissipation models have been proposed for use with the Timoshenko and Bernoulli-Euler beam models. A summary of the most frequently cited models may be found in Piché[26] and Wie and Bryson.[27] Experimental data is scant, particularly in the high-frequency range, so that none of these models can be considered definitive. Recent experimental work by Russell has demonstrated that the usual conjecture of "damping proportional to frequency" for internal (material) dissipation is *not* complete.[8] However, it is reasonably representative of material damping for certain low-to-midrange frequencies. Moreover, some of the more popular models were never intended for use in general transient analysis and fail to yield well-posed dynamical models (Chen and Russell[28]). One approach to developing dynamical models that include dissipation is to augment the variational development of the equations of motion by introducing a Rayleigh dissipation function (see also Russell[8]). We formulate such a function based on the following assumptions: 1) external dissipation forces are proportional to the coordinate velocities (i.e., $\dot\eta$, $\dot\phi$), and 2) internal dissipation forces are proportional to strain rates (i.e., shear strain rate $\dot\gamma = \partial\dot\eta/\partial z - \dot\phi$ and compressive strain rate $\partial\dot\phi/\partial z$). Thus, we define the dissipation function as

$$R(\dot\eta,\dot\phi) \equiv \frac{1}{2}\int_0^L \left\{c_1\left(\frac{\partial\eta}{\partial t}\right)^2 + c_2\left(\frac{\partial\phi}{\partial t}\right)^2 + c_3\left[\frac{\partial}{\partial z}\left(\frac{\partial\eta}{\partial t}\right) - \frac{\partial\phi}{\partial t}\right]^2 + c_4\left(\frac{\partial^2\phi}{\partial z\partial t}\right)^2 \right\} \mathrm{d}z$$

where c_1,c_2 are coefficients of external damping and c_3,c_4 are coefficients of internal (material) damping.

The modified Timoshenko equations are found after carrying out the usual variational calculations[25] yielding

$$\rho A \frac{\partial^2\eta}{\partial^2 t} = \frac{\partial}{\partial z}\left[\kappa GA\left(\frac{\partial\eta}{\partial z} - \phi\right)\right] - c_1\frac{\partial\eta}{\partial t} + c_3\left(\frac{\partial^3\eta}{\partial z^2\partial t} - \frac{\partial^2\phi}{\partial z\partial t}\right) \tag{39}$$

$$\rho I \frac{\partial^2\phi}{\partial t^2} = \frac{\partial}{\partial z}\left[EI\frac{\partial\phi}{\partial z}\right] + \kappa GA\left(\frac{\partial\eta}{\partial z} - \phi\right) - c_2\frac{\partial\phi}{\partial t} + c_3\left(\frac{\partial^2\eta}{\partial z\partial t} - \frac{\partial\phi}{\partial t}\right) + c_4\frac{\partial^3\phi}{\partial z^2\partial t} \tag{40}$$

To obtain the Bernoulli-Euler model we invoke the approximations $\rho I \to 0$ and $\partial\eta/\partial z - \phi \to 0$, with the following result:

$$\rho A\ddot\eta + c_4\frac{\partial^4\dot\eta}{\partial z^4} - c_2\frac{\partial^2\dot\eta}{\partial z^2} + c_1\dot\eta + EI\frac{\partial^4\eta}{\partial z^4} = 0 \tag{41}$$

Table 1 summarizes the dissipation terms appearing in Eq. (41) in terms of standard damping mechanisms.[28,29]

Individually, the damping terms appearing in Eq. (41) can produce distinct recognizable spectral patterns for the underlying infinitesimal generator A as constructed in Eqs. (7) and (8) and under certain (rather special) boundary conditions can provide well-defined, causal, stable systems.[7,28] The viscous term can provide a model for which the spectrum translates parallel to the real axis. The "square-root" damping term (from a mathematical model construction originally proposed by Chen and Russell[28]) can provide the "wedge" pattern characteristic of material dissipation. (The "square root" damping terminology is borrowed loosely here from the development in Chen and Russell[28] with respect to the specific boundary conditions, as discussed in the reference.) Kelvin-Voight damping can provide a spectral pattern where the sequence of eigenvalues have an accumulation point located on the finite real axis[7] (see, for example, Refs. 1, 7, 26–28). Our interest in the variational argument given here arises due to its generality and as a vehicle for a unifying analysis that provides qualitative insight as to the physical interpretation of these mathematical models. Of course, the appropriateness of these terms as models for damping must be considered on a case-by-case basis, depending on the problem at hand.

Similarly, from Eqs. (39) and (40), we obtain the string model with damping:

$$\rho A\frac{\partial^2\eta}{\partial t^2} = \frac{\partial}{\partial z}\left[\kappa GA\frac{\partial\eta}{\partial z}\right] - c_1\frac{\partial\eta}{\partial t} + c_3\frac{\partial^3\eta}{\partial z^2\partial t} \tag{42}$$

Equations (39) and (40) are easily put in the first-order form appropriate for our standard control modeling problem. For the Timoshenko model, introduce the variables $\nu(t,z)$, $\gamma(t,z)$ and write the equations as

$$\frac{\partial\eta}{\partial t} = \frac{\partial\nu}{\partial z} + \frac{c_3}{\rho A}\frac{\partial^2\eta}{\partial z^2} - \frac{c_3}{\rho A}\frac{\partial\phi}{\partial z} - c_1\eta \tag{43a}$$

$$\frac{\partial\nu}{\partial t} = \frac{\kappa G}{\rho}\left(\frac{\partial\eta}{\partial z} - \phi\right) \tag{43b}$$

Table 1 Dissipation terms and standard damping mechanisms

Term in eq. (41)	Form of damping
$c_4\frac{\partial^4}{\partial z^4}\dot\eta$	Kelvin-Voight
$-c_2\frac{\partial^4}{\partial z^4}\dot\eta$	Chen-Russell (or 'square root')
$c_1\dot\eta$	viscous

$$\frac{\partial \phi}{\partial t} = \frac{\partial \gamma}{\partial z} + \frac{A}{I}\nu + \frac{c_4}{\rho I}\frac{\partial^2 \phi}{\partial z^2} + \frac{c_3}{\rho I}\frac{\partial \eta}{\partial t} - \frac{(c_2 + c_3)}{\rho I}\phi \tag{43c}$$

$$\frac{\partial \gamma}{\partial t} = \frac{E}{\rho}\frac{\partial \phi}{\partial z} \tag{43d}$$

For the Bernoulli-Euler model of Eq. (41), introduce the variable $\gamma(z,t)$ and write as

$$\frac{\partial \eta}{\partial t} = -\frac{I}{A}\frac{\partial^2 \gamma}{\partial z^2} - \frac{c_4}{\rho A}\frac{\partial^4 \eta}{\partial z^4} + \frac{c_2}{\rho A}\frac{\partial^2 \eta}{\partial z^2} - c_1\eta \tag{44a}$$

$$\frac{\partial \gamma}{\partial t} = \frac{E}{\rho}\frac{\partial^2 \eta}{\partial z^2} \tag{44b}$$

For the string model of Eq. (42), introduce the variable $\nu(t,z)$ and write Eq. (42) in the first-order form:

$$\frac{\partial \eta}{\partial t} = \frac{\partial \nu}{\partial z} + c_3\frac{\partial^2 \eta}{\partial z^2} - c_1\eta \tag{45a}$$

$$\frac{\partial \nu}{\partial t} = \frac{\kappa G}{\rho}\frac{\partial \eta}{\partial z} \tag{45b}$$

III. Frequency Response Calculations for Distributed Parameter Systems

In this section we will be concerned with the computation of certain irrational transfer functions and a resolvent operator. This provides a complete model, including transient response for the distributed parameter system. For our purposes, the resolvent can be considered as an integral operator with its kernel a Green's function. Using transform methods, we compute explicit formulas for the abstract objects discussed previously. We focus on hyperbolic and parabolic linear (one-dimensional) structural models for distributed elements. As noted previously, such models can be used for elastic dynamics of beams, cables, etc. Finally, the computations are extended to hybrid system models consisting of interconnections of elastic components with rigid bodies and other lumped parameter models.

A. Hyperbolic Models

Consider a class of elastic structures represented by hyperbolic partial differential equations in one space dimension $0 \le z \le L$ (e.g., arising from models such as those discussed in the previous section):

$$\frac{\partial x(t,z)}{\partial t} = F\frac{\partial x(t,z)}{\partial z} + Hx(t,z) + Ev(t,z) \tag{46}$$

subject to boundary conditions

$$\Sigma_1 x(t,0) + \Gamma_1 x(t,L) = Df(t) \tag{47}$$

and initial conditions

$$x(0,z) = x^0(z) \in \mathcal{H}^n(0,L) \tag{48}$$

Here, x is an n-vector-valued state $x \in \mathcal{H}^n(0,L)$, $v \in \mathcal{H}^\ell(0,L)$ is an ℓ-vector-valued distributed disturbance, f is m-vector-valued boundary interactions, F and H are real $n \times n$ matrices with F nonsingular and diagonalizable,[18] and Σ_1, Γ_1 are $n \times n$ matrices. Controllability questions for systems of this type are considered in Russell.[13] After taking Laplace transforms in the temporal variable t, we obtain

$$\hat{X}(s,z) = \int_0^L G_r(s,z,w)\hat{M}(s,w)\,\mathrm{d}w + H_{BC}(s,z)\hat{F}(s) \tag{49}$$

where

$$\hat{M}(s,z) = x^0(\mathrm{w}) - C\hat{V}(s,w)$$

and $\hat{X}$, $\hat{V}$, $\hat{F}$ are the Laplace transforms of x, v, and f, respectively. The function $G_r(s,z,w)$ is the Green's function[2,30] for Eqs. (46) and (47), and $H_{BC}(s,z)$ is a transfer function from boundary interactions to state. Since in most cases of practical interest the control of flexible structures will be effected by actuators whose influence functions are highly localized, we have formulated our model with boundary control only.

Comparison with Eq. (1) clearly shows that the resolvent for the operator A: $\mathcal{H}^n(0,L) \to \mathcal{H}^n(0,L)$, defined by Eqs. (46) and (47), is the integral operator $\int_0^L G_r(s,z,w)\cdot \mathrm{d}w$. A straightforward calculation leads to the following form for H_{BC}:

$$H_{BC}(s,z) = N(s,z)D \tag{50}$$

where

$$N(s,z) = \Phi(s,z)\,[\Sigma_1 + \Gamma_1\Phi(s,L)]^{-1} \tag{51a}$$

$$\Phi(s,z) = e^{[F-1(sI-H)z]} \tag{51b}$$

The Green's function for Eqs. (46) and (47) is the solution to

$$\partial G_r(s,z,w)/\partial z = F^{-1}[sI - H]\,G_r(s,z,w) + I_n\delta(z - w) \tag{52}$$

subject to the boundary conditions

$$\Sigma_1 G_r(s,0,w) + \Gamma_1 G_r(s,L,w) = 0 \tag{53}$$

where $\delta(\cdot)$ is the Dirac delta function.[2,30] From Eq. (52) we see that the solution is discontinuous at the point $z = w$. After some computation, we can write

$$G_r(s,z,w) = \begin{cases} G_{r\,\mathrm{LEFT}}(s,z,w), & \text{for } 0 \le z \le w \\ G_{r\,\mathrm{RIGHT}}(s,z,w), & \text{for } w \le z \le L \end{cases} \tag{54}$$

with

$$G_{r\,\mathrm{LEFT}}(s,z,w) = -N(s,z)\Gamma_1\Phi(s,L - w) \tag{55}$$

$$G_{r\,\mathrm{RIGHT}}(s,z,w) = N(s,z)\Sigma_1\Phi(s, -w) \tag{56}$$

B. Parabolic Models

We begin with the normal-form, first-order model consisting of n equations as

$$\frac{\partial x(t,z)}{\partial t} = G\frac{\partial^2 x(t,z)}{\partial z^2} + F\frac{\partial x(t,z)}{\partial z} + Hx(t,z) + Ev(t,z) \tag{57}$$

subject to $2n$ boundary conditions

$$\Sigma_1 x(t,0) + \Sigma_2\frac{\partial x(t,0)}{\partial z} + \Gamma_1 x(t,L) + \Gamma_2\frac{\partial x(t,L)}{\partial z} = Df(t) \tag{58}$$

and n initial conditions

$$x(0,z) = x^0(z) \in \mathcal{H}^n(0,L) \tag{59}$$

Let $\Sigma = [\Sigma_1,\Sigma_2]$ and $\Gamma = [\Gamma_1,\Gamma_2]$—each $2n \times 2n$ real matrices.

Following the outlined procedure, the Green's function and boundary transfer function can be computed as follows. Let

$$\Lambda(s) = \begin{bmatrix} 0_n & I_n \\ -G^{-1}(H - sI_n) & -G^{-1}F \end{bmatrix} \tag{60a}$$

$$\Phi(s,z) = e^{\Lambda(s)z} \tag{60b}$$

$$M(s,z) = [I_n, 0]\Phi(s,z)[\Sigma + \Gamma\Phi(s,L)]^{-1} \tag{61}$$

Then the boundary transfer function is

$$H_{BC}(s,z) = M(s,z)D \tag{62}$$

and the Green's function is given by

$$G_r(s,z,w) = \begin{cases} G_{r\,\mathrm{LEFT}}(s,z,w), & \text{for } 0 \le z \le w \\ G_{r\,\mathrm{RIGHT}}(s,z,w), & \text{for } w \le z \le L \end{cases} \tag{63}$$

with

$$G_{r\,\mathrm{LEFT}} = -M(s,z)\Gamma\Phi(s,L-w)\begin{bmatrix} 0 \\ I_n \end{bmatrix} \tag{64}$$

$$G_{r\,\mathrm{RIGHT}} = M(s,z)\Sigma\Phi(s,-w)\begin{bmatrix} 0 \\ I_n \end{bmatrix} \tag{65}$$

C. Modeling of Hybrid Systems

In most applications, models for the dynamics of flexible structures involve interaction between various elastic and rigid components. In the particular case of flexible structures associated with large space structures, the potential topological configurations can be quite complex. Various elements such as beams, truss structures, cables, membranes, etc., may have dominant distributed parameter effects. Typically, a central body or bodies represent large concentrations of mass with respect to the overall low mass density of the flexible structure. These are most effectively represented by lumped parameter models of their rigid-body dynamics. Additionally, various attitude control actuators can add concentrated inertia elements that can be effectively modeled as lumped systems. Thus, carefully chosen linear hybrid models can provide an effective tool for the analysis of dynamics of vibrations and their effect on small angle motions for complex space platforms. In this section we consider the structures and computations of certain resulting transfer functions and the resolvent operator for the composite system along the lines of Sec. I.

The concept of a mechanical impedance (terminology borrowed from electrical network theory) has been used in structural dynamic modeling for many years.[29] The dynamic stiffness method (application to space structure modeling is reviewed in Piché[26]) uses this notion to compute effective transfer function models for interconnected structures.[31] Our approach here will follow along similar lines except that we will focus on computing the resolvent operator for a hybrid structure by direct manipulation of its kernel; viz, a Green's function.[2] A hybrid state-space model is constructed in Burns and Cliff[10] (where considerations are given for approximation and computation in the hybrid state-space). We will consider a hybrid state-space as consisting of a direct sum of spaces $\mathfrak{X}_\ell = \mathfrak{X}_\ell \oplus \mathfrak{X}_d$ where $\mathfrak{X}_d = \mathcal{H}^{N_d}$ is the distributed part constructed on an appropriate Hilbert space of N_d-vector-valued functions for a distributed parameter system (DPS) written in the abstract form of Eq. (7) and $\mathfrak{X}_\ell \equiv \boldsymbol{R}^{N_\ell}$, a finite-dimensional state-space of the lumped parameter system (LPS).

For control of hybrid structures, we restrict attention to the DPS modeled as either the hyperbolic or parabolic (or mixed) cases that, as we have seen, can be expressed in the frequency domain in the form

$$\hat{X}_d(s,z) = \int_0^L G_r(s,z,w)\hat{M}(s,w)\,\mathrm{d}w + H_{BC}(s,q)\hat{F}_d(s) \tag{66}$$

where

$$\hat{M}(s,w) = x_d^0(w) - E\hat{V}(s,w) \tag{67}$$

Clearly, Eqs. (66) and (67) can represent a disjoint collection of distributed elements such as beams, cables, etc. [Conceptually, a version of Eq. (66) also can be written for higher-dimensional spatial domains, but we feel for the current presentation that the required complexity of notation can mask the simplicity of the underlying concepts—see Butkovskiy[2] for details.]

All LPS component models are combined into a LPS state-space model as

$$\dot{x}_\ell(t) = A_\ell x_\ell(t) + B_\ell f_\ell(t),\; x_\ell^0 = x_\ell(0) \tag{68}$$

with $x_\ell \in R^{N_\ell} \equiv \mathfrak{X}_\ell$ a finite-dimensional real space. By taking Laplace transforms in Eq. (68), we write [analogous to Eq. (66)]

$$\hat{X}_\ell(s) = R_\ell(s)x_\ell^0 + H_\ell(s)\hat{F}_\ell(s) \tag{69}$$

where $R_\ell(s) = [sI_{N_\ell} - A_\ell]^{-1}$ is the resolvent for the (matrix) operator A_ℓ and $H_\ell(s) = R_\ell(s)B$.

The hybrid state space $\mathfrak{X} = \mathfrak{X}_\ell \oplus \mathfrak{X}_d$ consists of elements

$$x(t,z) = \begin{pmatrix} x_\ell(t) \\ x_d(t,z) \end{pmatrix} \tag{70}$$

which are $N = N_d + N_\ell$-valued functions of $z \in [0,L]$, $t > 0$. Finally, the interconnection of component systems is resolved through a topological constraint relation consisting of $m = m_d + m_\ell$ linear equations:

$$f(t) + T_1 x_d(t,0) + T_2 x_d(t,L) + T_3 x_\ell(t) = Ku(t) \tag{71}$$

where $u(t)$ is a k vector of control inputs to the hybrid system, T_1 and T_2 are $m \times N_d$, T_3 is $m \times N_\ell$, and K is $m \times k$ real mat rices. The hybrid modeling problem is to find an equation of the form of Eq. (66) by solving Eqs. (66) and (69–71) simultaneously for the hybrid state $x(t,z)$. We provide the resulting model in the following form:

$$\hat{X}(s,z) = \int_0^L \tilde{G}_r(s,z,w)\hat{M}(s,w)\,\mathrm{d}w + \tilde{R}(s,z)x_\ell^0 + \tilde{H}_{BC}(s,z)\hat{U}(s) \tag{72}$$

where $\hat{M}(s,w)$ is given in Eq. (67). The resolvent operator for the hybrid system is

$$R(s;A) = \left[\tilde{R}(s,z),\; \int_0^L \tilde{G}_r(s,z,w)\cdot \mathrm{d}w\right] \tag{73}$$

where $R(s;A): \mathfrak{X} \to D(A) \subseteq \mathfrak{X}$, $\tilde{G}_r$ and $\tilde{R}$, $N \times N_d$ and $N \times N_\ell$ respectively, are matrix-valued functions that can be computed explicitly as follows:

$$\tilde{R}(s,z) = \begin{bmatrix} I_{N_\ell} - H_\ell(s)\tilde{Q}_1(s) \\ -H_{BC}(s,z)\tilde{Q}_2(s) \end{bmatrix} T_3 R_\ell(s) \tag{74}$$

$$\tilde{G}_r(s,z,w) = \begin{bmatrix} -H_\ell(s)\tilde{Q}_1(s) \\ G_r(s,z,w) - H_{BC}(s,z)\tilde{Q}_2(s) \end{bmatrix} P(s,w) \tag{75}$$

where

$$\tilde{Q}(s) = [I_M + Q(s)]^{-1} = \begin{bmatrix} \tilde{Q}_1(s) \\ \tilde{Q}_2(s) \end{bmatrix} \tag{76}$$

$$Q(s) = [T_3 H_\ell(s),\; T_1 H_{BC}(s,0) + T_2 H_{BC}(s,L)] \tag{77}$$

$$P(s,w) = T_1 G_r(s,0,w) + T_2 G_r(s,L,w) \tag{78}$$

Finally, the $N \times k$ transfer function matrix from boundary control to hybrid state is

$$\tilde{H}(s,z) = \begin{bmatrix} H_\ell(s) & 0 \\ 0 & H_{BC}(s,z) \end{bmatrix} \tilde{Q}(s)K \tag{79}$$

The derivation of Eqs. (72–78) is straightforward and proceeds as follows. Substitute Eqs. (66) and (68) into Eq. (71) and solve for the interconnecting force $\hat{F}(s)$. This identifies the terms $Q(s)$, $P(s,w)$. Now substitute the appropriate components of $\hat{F}(s)$ into Eqs. (66) and (68) and use the hybrid state model Eq. (70).

D. Example: Hybrid Modeling

Next, we consider a simple example of a hybrid model. The simplicity of this example provides a completely transparent illustration of the required computations and serves to demonstrate that closed-form expressions can be obtained by direct algebraic simplification of the computations outlined in the previous section. The example can be motivated by any one of a dozen problems associated with the analysis of so-called *cantilevered modes* of vibration for a satellite with flexible appendage (e.g., the SCOLE problem Taylor[32]). Specifically, we consider the generic problem of a cantilevered beam with tip mass (see Fig. 1).

The equations of motion are determined from a standard variational approach. For this simple example we will use the approximation for a long thin beam (discussed in Sec. II as the "string" model) in which the Timoshenko equations discussed are reduced by neglecting the rotation angle of cross section $\phi = 0$ in Eqs. (20). The model for the hybrid system consists of Eq. (35) for $0 \le z \le L$, subject to the following boundary conditions at $z = L$:

$$m_L \frac{\partial^2 \eta(t,L)}{\partial t^2} + \kappa G A \frac{\partial \gamma(t,L)}{\partial z} = f_u(t) \tag{80}$$

and at $z = 0$:

$$\eta(t,0) = \frac{\partial \eta(t,0)}{\partial z} = 0 \tag{81}$$

Now Eq. (35) can be written in the form of Eqs. (66) and (67) by a particular choice of distributed state: $x_d(t,z) = [\gamma, \eta]^T$, where Eq. (35) becomes, with $\alpha^2 = \kappa G/\rho$,

$$\frac{\partial \eta(t,z)}{\partial t} = \alpha \frac{\partial \gamma(t,z)}{\partial z} \tag{82a}$$

$$\frac{\partial \gamma(t,z)}{\partial t} = \alpha \frac{\partial \eta(t,z)}{\partial z} \tag{82b}$$

Thus, we write the DPS in the canonical first-order form of Eqs. (66) and (67):

$$\frac{\partial x_d(t,z)}{\partial t} = \frac{\partial}{\partial z} \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} x_d(t,z) \tag{83}$$

subject to

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_d(t,0) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x_d(t,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} f_d(t) \tag{84}$$

To obtain a particular state-space model for the lumped system, we take the LPS state as $x_\ell = [x_{\ell 1}, \eta(t,L)]^T$, where the first coordinate is chosen to satisfy $\dot\eta(t,L) = x_{\ell 1}(t) - (\kappa GA/\alpha)\,\gamma(t,L)$. The LPS model can then be written as

$$\dot x_\ell(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_\ell(t) + \begin{bmatrix} 0 & 1/m_L \\ -\beta/\alpha m_L & 0 \end{bmatrix} \begin{bmatrix} f_{1\ell} \\ f_{2\ell} \end{bmatrix} \tag{85}$$

where $\beta = \kappa GA$. Finally, the topological interconnection is resolved by an equation of the form of Eq. (71):

$$\begin{bmatrix} f_{1\ell} \\ f_{2\ell} \\ f_d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} x_d(t,L) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} x_\ell(t) = \begin{bmatrix} f_u(t) \\ 0 \\ 0 \end{bmatrix}$$

We remark that for this example it is convenient (and relatively straightforward) to choose the state coordinates for each model so that the interactions at their boundaries are simple. This provides some insight into the meaning of the individual state variables with respect to the hybrid model, as shown in Fig. 2.

From Eqs. (81) and (82), we obtain the DPS Green's function of Eq. (66) using Eqs. (50–56), which can be simplified to the following:

$$G_{r\,\mathrm{LEFT}} = \frac{\begin{bmatrix} \sinh(s/\alpha)(w+z-L) + \sinh(s/\alpha)(w-z-L) & -\cosh(s/\alpha)(w-z-L) - \cosh(s/\alpha)(w+z-L) \\ -\cosh(s/\alpha)(w-z+L) + \cosh(s/\alpha)(w+z-L) & \sinh(s/\alpha)(w-z-L) - \sinh(s/\alpha)(w+z-L) \end{bmatrix}}{2\sinh(sL/\alpha)}$$

$$G_{r\,\mathrm{RIGHT}} = \frac{\begin{bmatrix} \sinh(s/\alpha)(w-z+L) + \sinh(s/\alpha)(w+z-L) & -\cosh(s/\alpha)(w-z+L) + \cosh(s/\alpha)(w+z-L) \\ -\cosh(s/\alpha)(w-z+L) + \cosh(s/\alpha)(w+z-L) & \sinh(s/\alpha)(w-z+L) - \sinh(s/\alpha)(w+z-L) \end{bmatrix}}{2\sinh(sL/\alpha)}$$

with

$$G_r(s,z,w) = \begin{cases} G_{r\,\mathrm{LEFT}}, & 0 \le z \le w \\ G_{r\,\mathrm{RIGHT}}, & w \le z \le L \end{cases}$$

$$H_{BC}(s,z) = \frac{\begin{bmatrix} \cosh(sz/\alpha) \\ \sinh(sz/\alpha) \end{bmatrix}}{\sinh(sL/\alpha)}$$

The final hybrid model can be written in terms of Eq. (72) by identifying the following terms. From Eq. (75), the hybrid Green's function G_r can be expressed using Eq. (75) given the terms

$$-H_\ell \tilde Q_1 P = \frac{\begin{bmatrix} -s\sinh(sw/\alpha) & s\cosh(sw/\alpha) \\ -\sinh(sw/\alpha) & \cosh(sw/\alpha) \end{bmatrix}}{\cosh(sL/\alpha) - m_L s^2 \sinh(sL/\alpha)}$$

$$-H_{BC}\tilde Q_2 P = \frac{\begin{bmatrix} -\sinh(s/\alpha)(w-z) - \sinh(s/\alpha)(w+z) & \cosh(s/\alpha)(w-z) + \cosh(s/\alpha)(w+z) \\ \cosh(s/\alpha)(w-z) - \cosh(s/\alpha)(w+z) & -\sinh(s/\alpha)(w-z) + \sinh(s/\alpha)(w+z) \end{bmatrix}}{2\sinh(sL/\alpha)[\cosh(sL/\alpha) - m_L s^2 \sinh(sL/\alpha)]}$$

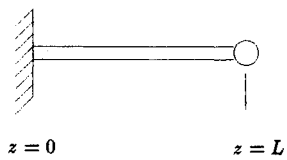

Fig. 1 Cantilevered beam with tip mass.

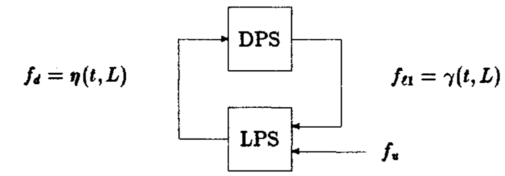

Fig. 2 Hybrid interconnection model.

From Eq. (74), we can write

$$\tilde{R}(s,z) = \frac{\begin{bmatrix} -m_L s \sinh(sL/\alpha) & -\cosh(sL/\alpha) \\ -m_L \sinh(sL/\alpha) & -m_L s \cosh(sL/\alpha) \\ -m_L \cosh(sL/\alpha) & -m_L s \cosh(sL/\alpha) \\ -m_L \sinh(sL/\alpha) & -m_L s \cosh(sL/\alpha) \end{bmatrix}}{\cosh(sL/\alpha) - m_L s^2 \sinh(sL/\alpha)}$$

Finally, from Eq. (77),

$$\tilde{H}(s,z) = \frac{\begin{bmatrix} -s \cosh(sL/\alpha) \\ -\cosh(sL/\alpha) \\ -m_L s^2 \cosh(sz/\alpha) \\ -m_L s \sinh(sz/\alpha) \end{bmatrix}}{\cosh(sL/\alpha) - m_L s^2 \sinh(sL/\alpha)}$$

These calculations were carried out using the computer algebra system SMP. Considerable algebraic reduction was necessary to achieve the final forms given, but the computer algebra program was flexible enough to permit programming for automatic simplification.

IV. Conclusions

Wiener-Hopf methods for control system design are *not* inherently limited to systems with rational transfer function models. Hence, they provide an opportunity for control synthesis by dealing directly with dynamical system models characterized by coupled systems of partial and ordinary differential equations. Nevertheless, the formulation of the required transfer functions and Green's functions is nontrivial even for relatively simple systems. The significance of such models is that the inevitable numerical approximations ultimately required for evaluating the model frequency responses can be addressed in the context of the control problem with respect for the control objective. We believe this approach can provide important insights for the design of control systems for flexible structures.

In this paper we have presented a systematic procedure for computing the required transfer function and Green's functions for systems composed of common elemental models used to describe flexible space structure dynamics. A key element of the procedure is the reduction of standard models for linear elastic elements (beams and rods) to well-posed state-space representations. We also have shown how these models may be combined with lumped element models in order to assemble models for the interaction of hybrid systems. Essential for the modeling computations are considerations of causality and well posedness.

Although systematic, the required calculations are tedious even for relatively simple structural systems. We have found it convenient to exploit a computer algebraic system (SMP was used throughout) for computations of all equations resulting in the example. The use of symbolic manipulation provides an appropriate environment for analysis as well as computation. For example, in the construction of hybrid models it is often important to check that the transfer function models for individual components are causal and proper. For the transcendental functions involved, a test for properness can be readily carried out by expanding the function in a Laurent series expansion about the point at infinity. This analysis can be readily carried out in SMP using the built-in power series function Ps[ ].

Perhaps of more significance is the potential for algebraic simplification of expressions and automatic generation of Fortran code for numerical evaluation of functions. For example, various numerical algorithms exist for the computation of modal frequencies from transfer function models.[26,31] The algorithm of Davis-Dickinson[15] provides a numerical method for computation of the matrix spectral factor central to control design. Both of these methods rely on accurate frequency response data, which can be obtained from the resulting transfer functions using finite-precision arithmetic only after some analysis of the functions involved. Through symbolic manipulation, a natural interface can be provided between model construction and numerical evaluation.[33]

Acknowledgments

This work was performed for the Air Force Office of Scientific Research under Contract F49620-84-C-0115. We would like to acknowledge the assistance of Mr. A. Virdy for careful computation and programming of the SMP code for the computations of the example.

References

[1]Balas, M. J., "Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams," *IEEE Transactions on Automatic Control,* Vol. AC-27, 1984, pp. 522–535.

[2]Butkovskiy, A. G., *Green's Functions and Transfer Functions Handbook,* Ellis Harwood Ltd., Chichester, England, 1982.

[3]Nicholson, J. W. and Bergman, L. A., "Free Vibration of Combined Dynamical Systems," *Journal of Engineering Mechanics,* ASC, Vol. 112, Jan. 1986, pp. 1–13.

[4]Nicholson, J. W. and Bergman, L. A., "On the Efficacy of the Modal Series Representation for the Green Functions of Vibrating Continuous Structures," *Journal of Sound and Vibration,* Vol. 98, No. 2, 1985, pp. 299–304.

[5]Cole, C. A. and Wolfram, S., "SMP—A Symbolic Manipulation Program," *Proceedings of the 1981 ACM Symposium on Symbolic Algebra Computation,* 1981, pp. 20–22.

[6]Wolfram, S., "Symbolic Mathematical Computation," *Communications of ACM,* Vol. 28, No. 4, 1985.

[7]Gibson, J., "An Analysis of Optimal Modal Regulation: Convergence and Stability," *SIAM Journal of Control and Optimization,* No. 5, 1981, pp. 686–707.

[8]Russell, D. L., "On Mathematical Models for the Elastic Beam with Frequency-Proportional Damping," *Frontiers in Applied Mathematics, Society of Industrial & Applied Mathematics,* to be published, 1989.

[9]Balas, M. J., "Modal Control of Certain Flexible Dynamic Systems," *SIAM Journal of Control,* Vol. 16, 1978, pp. 450–462.

[10]Burns, J. A. and Cliff, E. M., "An Approximation Technique for the Control and Identification of Hybrid Systems," *Proceedings of the Third VPI&SU/AIAA Symposium on Control of Large Structures,* 1981, pp. 269–284.

[11]Agarwala, A. S., "Modeling and Simulation of Hyperbolic Distributed Systems Arising in Process Dynamics," Ph.D. Thesis, Drexel Univ., Philadelphia, PA, June 1984.

[12]Meirovitch, L. and Öz, H., "An Assessment of Methods for the Control Large Space Structures," *Proceedings of the Joint Automatic Control Conference,* 1979.

[13]Russell, D. L., "Controllability and Stabilizability Theory for Linear Partial Differential Equations: Recent Progress and Open Questions," *SIAM Review,* Vol. 20, No. 4, 1978, pp. 639–739.

[14]Davis, J. H. and Barry, B. M., "A Distributed Model for Stress Control in Multiple Locomotive Trains," *Applied Mathematics Optimization,* Vol. 3, 1977, pp. 163-190.

[15]Davis, J. H. and Dickinson, R., "Spectral Factorization by Optimal Gain Iteration," *Journal of Applied Mathematics,* Vol. 43, 1983, pp. 389-301.

[16]Brockett, R. W., *Finite Dimensional Linear Systems,* Wiley, New York, 1970.

[17]Bennett, W. H. and Barkakati, N., "FlexCAD: Prototype Software for Modeling and Control of Flexible Structures," *Proceedings of the Third Symposium on Computer-Aided Control System Design,* Sept. 1986, pp. 64-69.

[18]Zauderer, E., *Partial Differential Equations of Applied Mathematics,* Wiley, New York, 1983.

[19]Lyczkowski, R. W., Gidaspow, D., Solbrig, C. W., and Hughes, E. D., "Characteristics and Stability Analysis of Transient One-Dimensional Two-Phase Flow Equations and their Finite Difference Approximations," *Nuclear Science and Engineering,* Vol. 66, 1978, pp. 378-396.

[20]Sursock, J. P., "Causality Violation of Complex-Characteristic Two-Phase Flow Equations," *International Journal of Multiphase Flow,* Vol. 8, No. 3, 1982, pp. 291-295.

[21]John, F., "A Note on 'Improper' Problems in Partial Differential Equations," *Communications on Pure and Applied Mathematics,* Vol. 8, 1955, pp. 591-594.

[22]Lax, P. D., "On the Notion of Hyperbolicity," *Communications on Pure and Applied Mathematics,* Vol. 9, 1980, pp. 267-293.

[23]Friedly, J. C., *Dynamic Behavior of Processes,* Prentice-Hall, Englewood Cliffs, NJ, 1972.

[24]Lapidus, L. and Pinder, G. F., *Numerical Solution of Partial Differential Equations in Science and Engineering,* Wiley, New York, 1982.

[25]Crandall, S. H., Karnopp, D. C., Kurtz, J. E. F., and Pridmore-Brown, D. C., *Dynamics of Mechanical and Electromechanical Systems,* McGraw-Hill, New York, 1968.

[26]Piché, R., "Frequency-Domain Continuum Modeling and Control of Third-Generation Spacecraft," Department of Communications, Ottawa, Ontario, Canada, CR-410-03, 1985.

[27]Wie, B. and Bryson, A., "Modeling and Control of Flexible Space Structures," *Proceedings of the 3rd VPI&SU/AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft,* Virginia Polytechnical Institute and State University, Blacksburg, VA, 1981.

[28]Chen, G. and Russell, D., "A Mathematical Model for Linear Elastic Systems with Structural Damping," *Quarterly Journal of Applied Mathematics,* Vol. 39, No. 4, 1982, pp. 433-454.

[29]Kolousek, V., *Dynamics in Engineering Structures,* Butterworths, London, 1983.

[30]Stakgold, I., *Green's Functions and Boundary Value Problems,* Wiley, New York, 1979.

[31]Poelaert, D., "DISTEL: A Distributed Element Program for Dynamic Modeling and Response Analysis of Flexible Structures," *Proceedings of the 4th VPI&SU/AIAA Symposium of Large Structures,* Virginia Polytechnical Institute and State University, Blacksburg, VA, June 1983, pp. 319-338.

[32]*SCOLE Workshop Proceedings,* NASA Langley Research Center, Hampton, VA, Dec. 1984.

[33]Wang, P. S., "FINGER: A Symbolic System for Automatic Generation of Numerical Programs in Finite Element Analysis," *Journal of Symbolic Computation,* Vol. 2, 1986, pp. 305-316.